

A Riemann–Hilbert approach to Painlevé IV

Marius van der Put and Jaap Top

April 2012

Abstract

This paper applies methods of Van der Put and Van der Put–Saito to the fourth Painlevé equation. One obtains a Riemann–Hilbert correspondence between moduli spaces of rank two connections on \mathbb{P}^1 and moduli spaces for the monodromy data. The moduli spaces for these connections are identified with Okamoto–Painlevé varieties and the Painlevé property follows. For an explicit computation of the full group of Bäcklund transformations, rank three connections on \mathbb{P}^1 are introduced, inspired by the symmetric form for PIV as was studied by M. Noumi and Y. Yamada.¹

Introduction

In this paper we apply the methods of [vdP-Sa, vdP1, vdP2] to the fourth Painlevé equation. We refer only to a few items of the extensive literature on Okamoto–Painlevé varieties. More details on Stokes matrices and the analytic classification of singularities can be found in [vdP-Si].

The Riemann–Hilbert approach to the Painlevé equation PIV consists of the construction of a moduli space \mathcal{M} of connections on the projective line and a moduli space \mathcal{R} for the monodromy data. The Riemann–Hilbert morphism $RH : \mathcal{M} \rightarrow \mathcal{R}$ assigns to a connection its monodromy data. The fibres of RH , i.e., the isomonodromic families in \mathcal{M} , are parametrized by $t \in T = \mathbb{C}$. The explicit form of the fibres produces the solutions of PIV.

¹ MSC2000: 14D20, 14D22, 34M55 *keywords*: Moduli space for linear connections, Irregular singularities, Stokes matrices, Monodromy spaces, Isomonodromic deformations, Painlevé equations

$RH^{ext} : \mathcal{M}^+(\theta_0, \theta_\infty) \rightarrow \mathcal{R}^+(\theta_0, \theta_\infty) \times T$, the extended Riemann–Hilbert morphism, is an analytic isomorphism between rather subtle moduli spaces $\mathcal{M}^+(\theta_0, \theta_\infty)$ and $\mathcal{R}^+(\theta_0, \theta_\infty) \times T$, depending on parameters θ_0, θ_∞ and provided with a level structure (or parabolic structure). The Painlevé Property for PIV with parameters θ_0, θ_∞ follows from this as well as the identification of $\mathcal{M}^+(\theta_0, \theta_\infty)$ with an Okamoto–Painlevé variety. Formulas for Bäcklund transformations, rational and Riccati solutions for PIV are derived.

The construction of \mathcal{M} involves the choice of a set \mathbf{S} of differential modules over $\mathbb{C}(z)$. In the first part of this paper the ‘classical’ choice for \mathbf{S} is treated. The second choice for \mathbf{S} is inspired by the symmetric form for PIV [No, No-Y], studied by M. Noumi and Y. Yamada. This leads to a different construction of \mathcal{M}, \mathcal{R} and Okamoto–Painlevé varieties, treated in Section 3.

1 The classical choice for \mathbf{S} and \mathcal{R}

Let \mathbf{S} be the set of the isomorphism classes of the differential modules (M, δ_M) over $\mathbb{C}(z)$ (with $\delta_M(fm) = (z \frac{d}{dz} f)m + f\delta_M(m)$) having the properties: $\dim M = 2$; $\Lambda^2 M$ is trivial; $0, \infty$ are the singular points and the Katz invariants are $r(0) = 0$, $r(\infty) = 2$. The variable z is normalized such that the (generalized) eigenvalues at ∞ are $\pm(z^2 + \frac{t}{2}z)$. Finally, we exclude the case that M is a direct sum of two proper submodules since this situation does not produce solutions for PIV.

The *monodromy data* at ∞ are given by the matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a_3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a_4 \\ 0 & 1 \end{pmatrix}$$

with respect to a basis of the symbolic solution space $V(\infty)$ at $z = \infty$ corresponding to the direct sum expression $V(\infty) = V_{z^2 + \frac{t}{2}z} \oplus V_{-(z^2 + \frac{t}{2}z)} = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. The first matrix is the formal monodromy and the others are the four Stokes matrices. The topological monodromy top_∞ at $z = \infty$ (which equals the topological monodromy at $z = 0$) is the product of these matrices in this order. Further we exclude the case $a_1 = a_2 = a_3 = a_4 = 0$, since this corresponds to the direct sum situation. The monodromy data form a variety $\mathcal{A} := \mathbb{C}^* \times (\mathbb{C}^4 \setminus \{(0, 0, 0, 0)\})$.

The base change $e_1, e_2 \mapsto \lambda e_1, \lambda^{-1} e_2$ induces an action of \mathbb{G}_m on \mathcal{A} . The *monodromy space* \mathcal{R} is the quotient \mathcal{A}/\mathbb{G}_m . This quotient can be obtained

by gluing the subspaces $\mathcal{R}_j, j = 1, \dots, 4$ of \mathcal{A} , defined by $a_j = 1$.

We observe (see [vdP-Sa], Theorem 1.7) that the map $\mathbf{S} \rightarrow \mathcal{R} \times T$, which maps a module in \mathbf{S} to its monodromy data and value of $t \in T = \mathbb{C}$, is bijective.

The *parameter space* is $\mathcal{P} = \mathbb{C} \times \mathbb{C}^*$ and $\mathcal{R} \rightarrow \mathcal{P}$ maps an element of \mathcal{R} to $(\text{trace}(\text{top}_\infty), \alpha)$. The fibre above (s, α) is denoted by $\mathcal{R}[s, \alpha]$. This fibre is a smooth, connected surface for $s \neq \pm 2$. The fibre $\mathcal{R}[2, \alpha]$ has *one singular point* and this point corresponds to $\text{top}_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Similarly, $\mathcal{R}[-2, \alpha]$ has *one singular point* corresponding to $\text{top}_\infty = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The singular points are the reason for introducing a *level structure* (or ‘parabolic structure’ in the terminology of [In, IIS1, IIS2, IISA]). For the monodromy data this is a line $L \subset V(\infty)$ which is invariant under top_∞ . The new monodromy space is denoted by \mathcal{R}^+ . For a module M in \mathbf{S} the level structure is a 1-dimensional submodule N of $\mathbb{C}((z)) \otimes M$. The submodule N corresponds to an eigenvector of the topological monodromy top_0 at $z = 0$ (which is equal to top_∞). The new set is denoted by \mathbf{S}^+ . For the parameter space, the level structure is the introduction of an eigenvalue β of top_∞ . The new parameter space $\mathcal{P}^+ = \mathbb{C}^* \times \mathbb{C}^*$ maps to \mathcal{P} by $(\beta, \alpha) \mapsto (\beta + \beta^{-1}, \alpha)$.

The fibres of $\mathcal{R}^+ \rightarrow \mathcal{P}^+$ are denoted by $\mathcal{R}^+(\beta, \alpha)$. The morphism $\mathcal{R}^+(\beta, \alpha) \rightarrow \mathcal{R}[\beta + \beta^{-1}, \alpha]$ is an isomorphism for $\beta \neq \pm 1$. A computation shows that

Lemma 1.1 $\mathcal{R}^+(\pm 1, \alpha) \rightarrow \mathcal{R}[\pm 2, \alpha]$ is the minimal resolution.

The map $\mathbf{S}^+ \rightarrow \mathcal{P}^+$ is defined by $\beta = e^{2\pi i \lambda}$ where $\delta_M n = \lambda n$ for a basis vector n of $N \subset \mathbb{C}((z)) \otimes M$ and α as before. The fibre is written as $\mathbf{S}^+(\beta, \alpha)$.

Lemma 1.2 The map $\mathbf{S}^+(\beta, \alpha) \rightarrow \mathcal{R}^+(\beta, \alpha) \times T$ is bijective.

The *reducible locus* of \mathcal{R}^+ (i.e., the monodromy data is reducible) is the disjoint union of the closed sets defined (in the notation of \mathcal{A}) by $a_2 = a_4 = 0$ and $a_1 = a_3 = 0$. The space $\mathcal{R}^+(\beta, \alpha)$ contains no reducible elements for $\beta^{\pm 1} \neq \alpha$. If $\beta^{\pm 1} = \alpha$, then the reducible locus of $\mathcal{R}^+(\beta, \alpha)$ consists of two non intersecting projective lines.

2 The moduli space $\mathcal{M}(\theta_0, \theta_\infty)$

Choose θ_0 with $\beta = e^{\pi i \theta_0}$ and θ_∞ with $\alpha = e^{\pi i \theta_\infty}$. The aim is to replace the set $\mathbf{S}^+(\beta, \alpha)$ by a moduli space of connections $\mathcal{M}(\theta_0, \theta_\infty)$ and to study the extended Riemann–Hilbert map $RH^{ext} : \mathcal{M}(\theta_0, \theta_\infty) \rightarrow \mathcal{R}^+(\beta, \alpha) \times T$.

Let a module $(M, N) \in \mathbf{S}^+(\beta, \alpha)$ be given. We define a connection (\mathcal{W}, ∇) on the projective line with $\nabla : \mathcal{W} \rightarrow \Omega([0] + 3[\infty]) \otimes \mathcal{W}$, with generic fibre M , by prescribing the connection $D := \nabla_{z \frac{d}{dz}}$ locally at $z = 0$ as $z \frac{d}{dz} + \begin{pmatrix} \frac{\theta_0}{2} & * \\ 0 & -\frac{\theta_0}{2} \end{pmatrix}$ and locally at $z = \infty$ as $z \frac{d}{dz} + \begin{pmatrix} \omega & 0 \\ c & -\omega \end{pmatrix}$ with $\omega = z^2 + \frac{t}{2}z + \frac{\theta_\infty}{2}$. This is equivalently to choosing ‘invariant lattices’ at $z = 0$ and $z = \infty$. The invariant lattice at $z = 0$ is $\mathbb{C}[[z]]g_1 + \mathbb{C}[[z]]g_2 \subset \mathbb{C}((z)) \otimes M$ with $N = \mathbb{C}((z))g_1$ and the matrix of δ_M with respect to g_1, g_2 is $\begin{pmatrix} \frac{\theta_0}{2} & * \\ 0 & -\frac{\theta_0}{2} \end{pmatrix}$. The invariant lattice at $z = \infty$ is $\mathbb{C}[[z^{-1}]]h_1 + \mathbb{C}[[z^{-1}]]h_2 \subset \mathbb{C}((z^{-1})) \otimes M$ such that $\delta_M h_1 = \omega h_1$, $\delta_M h_2 = -\omega h_2$.

The second exterior power of (\mathcal{W}, ∇) is $d : \mathcal{O} \rightarrow \Omega$. Thus \mathcal{W} has degree 0 and type $\mathcal{O}(k) \oplus \mathcal{O}(-k)$ with $k \geq 0$. If M is irreducible, then $k \in \{0, 1\}$. The reducible modules are studied in Observations 2.3.

We consider the case $k \in \{0, 1\}$. The connection (\mathcal{V}, ∇) , defined by replacing the invariant lattice $\mathbb{C}[[z]]g_1 + \mathbb{C}[[z]]g_2$ by $\mathbb{C}[[z]]g_1 + \mathbb{C}[[z]]zg_2$, has type $\mathcal{O} \oplus \mathcal{O}(-1)$. Further we identify \mathcal{V} with $\mathcal{O}e_1 \oplus \mathcal{O}(-[0])e_2$.

2.1 The connections on $\mathcal{V} := \mathcal{O}e_1 + \mathcal{O}(-[0])e_2$

The connection $D = \nabla_{z \frac{d}{dz}} : \mathcal{V} \rightarrow \mathcal{O}(2[\infty]) \otimes \mathcal{V}$, obtained from $(M, N) \in \mathbf{S}^+(\beta, \alpha)$ and the prescribed invariant lattices, has, with respect to the basis e_1, e_2 , the matrix $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a = a_0 + a_1z + a_2z^2$, $b = b_{-1}z^{-1} + \dots + b_2z^2$, $c = c_1z + c_2z^2$. The local data at $z = \infty$ yields the equations

$$a_2^2 + b_2c_2 = 1, \quad 2a_1a_2 + b_2c_1 + b_1c_2 = t, \quad 2a_0a_2 + a_1^2 + b_1c_1 + b_0c_2 = \theta_\infty + \frac{t^2}{4}.$$

For $z = 0$ one obtains $a_0(a_0 - 1) + b_{-1}c_1 = \frac{\theta_0}{2}(\frac{\theta_0}{2} - 1)$.

As a start, we forget the level structure N of the pair $(M, N) \in \mathbf{S}^+(\beta, \alpha)$ and assume that $c_1z + c_2z^2 \neq 0$. The above variables a_*, b_*, c_*, t and equations define a space \mathcal{C} of dimension 6. We have to divide by the group G of transformations $e_1 \mapsto e_1$, $e_2 \mapsto \lambda e_2 + (x_0 + x_1z^{-1})e_1$ of \mathcal{V} . The quotient \mathcal{C}/G is by definition the moduli space $\mathcal{M}(\theta_0, \theta_\infty)$.

Proposition 2.1 $\mathcal{M}(\theta_0, \theta_\infty)$ is a good geometric quotient of \mathcal{C} in the sense that there exists a G -equivariant isomorphism $G \times \mathcal{M}(\theta_0, \theta_\infty) \rightarrow \mathcal{C}$. $\mathcal{M}(\theta_0, \theta_\infty)$ is smooth for $\theta_0 \neq 1$. For a connection $D \in \mathcal{M}(1, \theta_\infty)$, which is a singular point, there is a basis of $\widehat{\mathcal{V}}_0$ for which D has the form $z \frac{d}{dz} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

Proof. The ‘first standard form’ ST_1 is the closed subset of \mathcal{C} defined by: $z \frac{d}{dz} + \begin{pmatrix} a_2 z^2 & b \\ z + c_2 z^2 & -a_2 z^2 \end{pmatrix}$ with $b_2 = -c_2(\theta_\infty + \frac{t^2}{4} - b_0 c_2) + t$, $b_1 = \theta_\infty + \frac{t^2}{4} - b_0 c_2$, $b_{-1} = \frac{\theta_0}{2}(\frac{\theta_0}{2} - 1)$ and $a_2^2 + c_2 t - c_2^2(\theta_\infty + \frac{t^2}{4} - b_0 c_2) - 1 = 0$. The obvious morphism $G \times ST_1 \rightarrow \{(a_*, b_*, c_*) \in \mathcal{C} \mid c_1 \neq 0\}$ is an isomorphism. $\mathbb{C}[a_2, c_2, t, b_0]/(a_2^2 + c_2 t - c_2^2(\theta_\infty + \frac{t^2}{4} - b_0 c_2) - 1)$ is the coordinate ring of ST_1 and ST_1 is nonsingular.

The ‘second standard form’ ST_2 is the closed subset of \mathcal{C} defined by: $z \frac{d}{dz} + \begin{pmatrix} a_0 & b \\ c_1 z + z^2 & -a_0 \end{pmatrix}$ with $b = z^2 + b_1 z + b_0 + b_{-1} z^{-1}$, $c = c_1 z + z^2$ and $c_1 + b_1 = t$, $b_1 c_1 + b_0 = \theta_\infty + \frac{t^2}{4}$, $a_0(a_0 - 1) + b_{-1} c_1 = \frac{\theta_0}{2}(\frac{\theta_0}{2} - 1)$. The obvious morphism $G \times ST_2 \rightarrow \{(a_*, b_*, c_*) \in \mathcal{C} \mid c_2 \neq 0\}$ is an isomorphism. $\mathbb{C}[a_0, c_1, t, b_{-1}]/(a_0(a_0 - 1) + b_{-1} c_1 - \frac{\theta_0}{2}(\frac{\theta_0}{2} - 1))$ is the coordinate ring of ST_2 . For fixed t , one finds one singular point: $a_0 = 1/2$, $b_{-1} = c_1 = 0$, $\theta_0 = 1$.

In the above case, one easily verifies that D has the form $z \frac{d}{dz} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ w.r.t. a basis of $\widehat{\mathcal{V}}_0$. The quotient \mathcal{C}/G is obtained by gluing the two ‘charts’ ST_1 and ST_2 in the obvious way. \square

Observations 2.2 The level structure for $\mathcal{M}(\theta_0, \theta_\infty)$.

For a connection $D \in \mathcal{M}(\theta_0, \theta_\infty)$, the level structure is a 1-dimensional submodule $N \subset \mathbb{C}((z)) \otimes \widehat{\mathcal{V}}_0$ with a generator n such that $\delta n = e^{\pi i \theta_0} n$. The space $\mathcal{M}^+(\theta_0, \theta_\infty)$ denotes the addition of this level structure to $\mathcal{M}(\theta_0, \theta_\infty)$.

If top_0 , the topological monodromy at $z = 0$ of the connection D , is not $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then the level structure N is unique.

If $\text{top}_0 = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $\theta_0 \in \mathbb{Z}$. Further $\widehat{\mathcal{V}}_0$ has a basis v_1, v_2 for which D has the form $z \frac{d}{dz} + \begin{pmatrix} \frac{\theta_0}{2} & 0 \\ 0 & 1 - \frac{\theta_0}{2} \end{pmatrix}$. If $\frac{\theta_0}{2} \neq 1 - \frac{\theta_0}{2}$, the basis v_1, v_2 is unique up to multiplication by constants. Then one defines the level structure N by $N = \mathbb{C}((z))v_1$.

In the final case $\text{top}_0 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\theta_0 = 1$, the connection D does not prescribe a level structure. We replace $\mathcal{M}(1, \theta_\infty)$ by $\mathcal{M}^+(1, \theta_\infty)$ defined as the closed subspace of $\mathcal{M}(1, \theta_\infty) \times \mathbb{P}^1$ consisting of the equivalence classes of the tuples (D, L) with $D \in \mathcal{M}(1, \theta_\infty)$ and L a line in $\widehat{\mathcal{V}}_0$ at $z = 0$, invariant

under D . We will verify that (for fixed t) $\mathcal{M}^+(1, \theta_\infty) \rightarrow \mathcal{M}(1, \theta_\infty)$ is the minimal resolution.

Verification. The chart ST_2 of $\mathcal{M}(1, \theta_\infty)$ consists of the differential operators $z \frac{d}{dz} + \begin{pmatrix} a_0 & zb \\ c_1+z & 1-a_0 \end{pmatrix}$ with $b = z^2 + b_1z + b_0 + b_{-1}z^{-1}$, $c_1 + b_1 = t$, $b_1c_1 + b_0 = \theta_\infty + \frac{t^2}{4}$, $a_0(a_0 - 1) + b_{-1}c_1 = -\frac{1}{4}$. The line $L = \mathbb{C} \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}$ is generated by a nonzero element $\begin{pmatrix} x_1 \\ x_0 \end{pmatrix} \in \mathbb{C}[[z]]^2$ satisfying $\{z \frac{d}{dz} + \begin{pmatrix} a_0 & zb \\ c_1+z & 1-a_0 \end{pmatrix}\} \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}$. In the case $a_0 = \frac{1}{2}$, $b_{-1} = c_1 = 0$, the operator $z \frac{d}{dz} + \begin{pmatrix} a_0 & zb \\ c_1+z & 1-a_0 \end{pmatrix}$ is equivalent over $\mathbb{C}[[z]]$ to $z \frac{d}{dz} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. Thus the possible lines L form a projective line.

In the opposite case, the operator is equivalent over $\mathbb{C}[[z]]$ to $z \frac{d}{dz} + \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$ and there is only one L .

Observations 2.3 *The reducible locus of $\mathcal{M}(\theta_0, \theta_\infty)$.*

Put $\omega = z^2 + \frac{t}{2}z + \frac{\theta_\infty}{2}$ and let $c \in \mathbb{C}[z^{-1}, z]$. If a reducible connection is present in $\mathcal{M}(\theta_0, \theta_\infty)$, then $\frac{\theta_0}{2} \in \pm \frac{\theta_\infty}{2} + \mathbb{Z}$. There are two types of reducible modules in \mathbf{S}^+ . Type (1) is represented by $z \frac{d}{dz} + \begin{pmatrix} -\omega & 0 \\ c & \omega \end{pmatrix}$ and Type (2) is represented by $z \frac{d}{dz} + \begin{pmatrix} \omega & 0 \\ c & -\omega \end{pmatrix}$.

For a given reducible module M , say of type (1), one defines (as before) the connection (\mathcal{V}, D) with generic fibre M by the local operators $z \frac{d}{dz} + \begin{pmatrix} \frac{\theta_0}{2} & * \\ 0 & 1 - \frac{\theta_0}{2} \end{pmatrix}$ at $z = 0$ and $z \frac{d}{dz} + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$ at $z = \infty$. Assume that type (1) is not present in $\mathcal{M}(\theta_0, \theta_\infty)$. Then $\mathcal{V} \cong O(k) \oplus O(-k-1)$ with $k \geq 1$ and one identifies \mathcal{V} with $O(k[0])e_1 \oplus O(-(k+1)[0])e_2$. A computation of D in this case leads to two possible relations, namely $\frac{\theta_0}{2} = \frac{\theta_\infty}{2} - k$ or $1 - \frac{\theta_0}{2} = \frac{\theta_\infty}{2} - k$. Thus one finds the list for type (1). The list for type (2) is found in a similar way.

Type (1) is present in precisely the following cases:

$$\begin{aligned} \theta_0 &\geq \theta_\infty \text{ for } \frac{\theta_0}{2} \in \frac{\theta_\infty}{2} + \mathbb{Z} \text{ and } \frac{\theta_0}{2} \notin -\frac{\theta_\infty}{2} + \mathbb{Z} \\ \theta_0 &\leq -\theta_\infty + 2 \text{ for } \frac{\theta_0}{2} \notin \frac{\theta_\infty}{2} + \mathbb{Z} \text{ and } \frac{\theta_0}{2} \in -\frac{\theta_\infty}{2} + \mathbb{Z} \\ \theta_0 &\geq \theta_\infty \text{ or } \theta_0 \leq -\theta_\infty + 2 \text{ for } \frac{\theta_0}{2} \in \frac{\theta_\infty}{2} + \mathbb{Z} \text{ and } \frac{\theta_0}{2} \in -\frac{\theta_\infty}{2} + \mathbb{Z} \end{aligned}$$

Type (2) is present in precisely the following cases:

$$\begin{aligned} \theta_0 &\leq \theta_\infty + 2 \text{ for } \frac{\theta_0}{2} \in \frac{\theta_\infty}{2} + \mathbb{Z} \text{ and } \frac{\theta_0}{2} \notin -\frac{\theta_\infty}{2} + \mathbb{Z} \\ \theta_0 &\geq -\theta_\infty \text{ for } \frac{\theta_0}{2} \notin \frac{\theta_\infty}{2} + \mathbb{Z} \text{ and } \frac{\theta_0}{2} \in -\frac{\theta_\infty}{2} + \mathbb{Z} \\ \theta_0 &\leq \theta_\infty + 2 \text{ or } \theta_0 \geq -\theta_\infty \text{ for } \frac{\theta_0}{2} \in \frac{\theta_\infty}{2} + \mathbb{Z} \text{ and } \frac{\theta_0}{2} \in -\frac{\theta_\infty}{2} + \mathbb{Z} \end{aligned}$$

Examples: We use the notation $z\frac{d}{dz} + \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ of the space \mathcal{C} . Suppose $b = 0$. Then $a_2^2 = 1$, $2a_1a_2 = t$, $2a_0a_2 + a_1^2 = \theta_\infty + \frac{t^2}{4}$, $(a_0 - \frac{1}{2})^2 = (\frac{\theta_0}{2} - \frac{1}{2})^2$ and the non zero element $c_1z + c_2z^2$ is unique up to multiplication by a non zero constant. One finds in general four reducible families (with some overlap for $\theta_0 = 1$ and/or $\theta_\infty = \pm 1$):

$$\begin{aligned} b = 0, \quad a_2 = 1, \quad a_1 = \frac{t}{2}, \quad a_0 = \frac{\theta_\infty}{2}, \quad \frac{\theta_\infty}{2} = \frac{1}{2} \pm (\frac{\theta_0}{2} - \frac{1}{2}) \text{ and} \\ b = 0, \quad a_2 = -1, \quad a_1 = -\frac{t}{2}, \quad a_0 = -\frac{\theta_\infty}{2}, \quad -\frac{\theta_\infty}{2} = \frac{1}{2} \pm (\frac{\theta_0}{2} - \frac{1}{2}). \end{aligned}$$

We observe that these examples are precisely the cases of an equality sign in the lists for type (1) and type (2).

Proposition 2.4 *Put $\beta = e^{\pi i \theta_0}$, $\alpha = e^{\pi i \theta_\infty}$. Let $F : \mathcal{M}^+(\theta_0, \theta_\infty) \rightarrow \mathbf{S}^+(\beta, \alpha)$ be the map that sends a tuple (D, L) to (M, N) where M is the generic fibre of D and $N = \mathbb{C}((z)) \otimes L$. The map F is injective and its image contains the ‘irreducible locus’ of $\mathbf{S}^+(\beta, \alpha)$. A component of the ‘reducible locus’ lies in the image of F if and only if θ_0, θ_∞ satisfy the corresponding inequality of Observations 2.3.*

Proof. The injectivity of F follows from the construction of $\mathcal{M}^+(\theta_0, \theta_\infty)$. If M is irreducible then the vector bundle \mathcal{W} , introduced in the beginning of this section, has type $O(k) \oplus O(-k)$ with $k \in \{0, 1\}$. Therefore the subbundle \mathcal{V} has type $O \oplus O(-1)$ and can be identifies with $Oe_1 \oplus O(-[0])e_2$. Thus the image of F contains the ‘irreducible locus’ of $\mathbf{S}^+(\beta, \alpha)$. The final statement follows from Observations 2.3. \square

Define $\mathbf{S}^+(\theta_0, \theta_\infty) \subset \mathbf{S}^+(\beta, \alpha)$ (for $\beta = e^{\pi i \theta_0}, \alpha = e^{\pi i \theta_\infty}$) to be the image of F and let $\mathcal{R}^+(\theta_0, \theta_\infty) \subset \mathcal{R}^+(\beta, \alpha)$ be the corresponding open subset.

Corollary 2.5 *$RH^{ext} : \mathcal{M}^+(\theta_0, \theta_\infty) \rightarrow \mathcal{R}^+(\theta_0, \theta_\infty) \times T$, the extended Riemann–Hilbert map, is an analytic isomorphism.*

Proof. RH^{ext} is bijective since $\mathbf{S}^+(\theta_0, \theta_\infty) \rightarrow \mathcal{R}^+(\theta_0, \theta_\infty) \times T$ is bijective. The two spaces $\mathcal{M}^+(\theta_0, \theta_\infty)$ and $\mathcal{R}^+(\theta_0, \theta_\infty) \times T$ are smooth and so RH^{ext} is an analytic isomorphism (see [vdP1, vdP2]). \square

2.2 Isomonodromic families, Okamoto–Painlevé spaces

An isomonodromic family above the chart ST_2 of $\mathcal{M}^+(\theta_0, \theta_\infty)$ has the form $z\frac{d}{dz} + A$ with $A = \begin{pmatrix} a_0 & b \\ c & -a_0 \end{pmatrix}$ with $c = z^2 - qz$, $b = z^2 + b_1z + b_0 + b_{-1}z^{-1}$,

$b = z^2 + (t+q)z + q(t+q) + \theta_\infty + \frac{t^2}{4} + \left(\frac{(a_0 - \frac{1}{2})^2 - (\frac{\theta_0}{2} - \frac{1}{2})^2}{q}\right)z^{-1}$, where a_0 and q are functions of t . Isomonodromy is equivalent to the existence of an operator $\frac{d}{dt} + B$, commuting with $z\frac{d}{dz} + A$. In other terminology $z\frac{d}{dz} + A, \frac{d}{dt} + B$ is a *Lax pair*. This is equivalent to the equation $\frac{d}{dt}(A) = z\frac{d}{dz}(B) + [A, B]$. One observes that B has trace zero and that the entries of B have the form $d_{-1}(t)z + d_0(t) + d_1(t)z$. Using MAPLE one obtains the solution $B = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}$ with $B_1 = \frac{q(q+b_1)+b_0}{2}z^{-1} + \frac{b_1+q}{2} + \frac{1}{2}z$, $B_2 = \frac{1}{2}z$,

$$q' = a_0 - \frac{1}{2}, \quad a'_0 = \frac{b_{-1} + q(q(b_1 + q) + b_0)}{2} \text{ and the fourth Painlevé equation}$$

$$q'' = \frac{(q')^2}{2q} + \frac{3q^3}{2} + tq^2 + \frac{q}{8}(4\theta_\infty + t^2) - \frac{(\theta_0 - 1)^2}{8q} \text{ with parameters } \theta_0, \theta_\infty.$$

Isomonodromy for reducible families. An isomonodromic family of operators $z\frac{d}{dz} + \begin{pmatrix} \omega & 0 \\ z^2 - qz & -\omega \end{pmatrix}$, with $\omega = z^2 + \frac{t}{2}z + \frac{d}{2}$, commutes with an operator of the form $\frac{d}{dt} + \begin{pmatrix} \tau & 0 \\ \frac{\tau}{2} & -\tau \end{pmatrix}$. One computes that $\tau = \frac{2z+2q+t}{4}$ and $q' = q^2 + \frac{t}{2}q + \frac{d-1}{2}$. Then q is a Riccati solution of PIV with $d = \theta_\infty$ and $d = 1 \pm (\theta_0 - 1)$.

An isomonodromic family $z\frac{d}{dz} + \begin{pmatrix} -\omega & 0 \\ z^2 - qz & \omega \end{pmatrix}$ with $\omega = z^2 + \frac{t}{2}z + \frac{d}{2}$ produces the equation $q' = -q^2 - \frac{t}{2}q - \frac{d+1}{2}$. Then q is a Riccati solution of PIV with $d = \theta_\infty$ and $d = -1 \pm (\theta_0 - 1)$.

Observations 2.6 *The solutions q_r with $r \in \mathcal{R}^+(\theta_0, \theta_\infty)$.*

The fibre of $\mathcal{M}^+(\theta_0, \theta_\infty) \rightarrow \mathcal{R}^+(\theta_0, \theta_\infty)$ above r is, by Corollary 2.5, isomorphic to T . Write q_r for the function q appearing in the formula for the chart ST_2 . Then q_r is a meromorphic solution of PIV, defined on all of T .

Theorem 2.7 *The fourth Painlevé equation has the Painlevé property. The moduli space $\mathcal{M}^+(\theta_0, \theta_\infty)$ is analytically isomorphic to the Okamoto–Painlevé space for PIV with parameters θ_0, θ_∞ .*

Proof. Let a local solution Q of PIV with parameters θ_0, θ_∞ be given. Let U be an open disk, where Q is holomorphic and has no zeros. Consider the operator $z\frac{d}{dz} + \begin{pmatrix} \tilde{a}_0 & \tilde{b} \\ z^2 - Qz & -\tilde{a}_0 \end{pmatrix}$ with $\tilde{a}_0 = \frac{dQ}{dt} + \frac{1}{2}$, $\tilde{b} = z^2 + (t+Q)z + Q(t+Q) + \theta_\infty + \frac{t^2}{4} + \frac{(\tilde{a}_0 - \frac{1}{2})^2 - (\frac{\theta_0}{2} - \frac{1}{2})^2}{Q}z^{-1}$. This defines an analytic map $U \rightarrow \mathcal{M}^+(\theta_0, \theta_\infty)$. Since Q is a local solution of PIV, the map $U \rightarrow \mathcal{M}^+(\theta_0, \theta_\infty) \rightarrow \mathcal{R}^+(\theta_0, \theta_\infty)$ is constant. Let r be its image. Then Q coincides with q_r on U . Thus

Q extends to a global solution of PIV and this equation has the Painlevé property.

The bundle $\mathcal{M}^+(\theta_0, \theta_\infty) \rightarrow T$, with its foliation defined by the fibres of $\mathcal{M}^+(\theta_0, \theta_\infty) \rightarrow \mathcal{R}^+(\theta_0, \theta_\infty)$, is the Okamoto–Painlevé variety according to the isomorphism of Corollary 2.5. \square

We note that $\mathcal{R}^+(\theta_0, \theta_\infty)$ is the *space of initial conditions*.

2.3 $Aut(\mathbf{S}^+)$ and Bäcklund transformations

Natural automorphisms of \mathbf{S}^+ are:

- (1). $\sigma_1 : (M, N) \mapsto (M, N^*)$ where N^* is a submodule of $\mathbb{C}((z)) \otimes M$ such that $N \oplus N^* = \mathbb{C}((z)) \otimes M$. This is well defined for $\beta \neq \pm 1$. For $\beta = \pm 1$, the module N^* might not exist or might not be unique. It seems correct to define $N^* := N$ for $\beta = \pm 1$.
- (2). $\sigma_2 : (M, N) \mapsto (M \otimes A, N \otimes A)$, where $A = \mathbb{C}((z))a$ and $\delta a = \frac{1}{2}a$.
- (3). $\sigma_3 : (M, N) \mapsto \mathbb{C}(z) \otimes_\phi (M, N)$, where ϕ is the automorphism of $\mathbb{C}(z)$ which is the identity on \mathbb{C} and maps z to iz . Let $Aut(\mathbf{S}^+)$ denote the group generated by σ_j , $j = 1, 2, 3$.

	β	α	t	z
σ_1	β^{-1}	α	t	z
σ_2	$-\beta$	$-\alpha$	t	z
σ_3	β	α^{-1}	it	iz

The above group is commutative and has order 16. The following table is a choice of lifting the generators to actions on $\theta_0, \theta_\infty, t, z$.

	θ_0	θ_∞	t	z
$\tilde{\sigma}_1$	$2 - \theta_0$	θ_∞	t	z
$\tilde{\sigma}_2$	$\theta_0 + 1$	$\theta_\infty + 1$	t	z
$\tilde{\sigma}_3$	θ_0	$-\theta_\infty$	it	iz

The induced morphisms $\tilde{\sigma}_1 : \mathcal{M}^+(\theta_0, \theta_\infty) \rightarrow \mathcal{M}^+(-\theta_0 + 2, \theta_\infty)$ and $\tilde{\sigma}_3 : \mathcal{M}^+(\theta_0, \theta_\infty) \rightarrow \mathcal{M}^+(\theta_0, -\theta_\infty)$ are evident from the standard operators representing the points of $\mathcal{M}^+(\theta_0, \theta_\infty)$. A MAPLE computation yields the explicit morphisms $\tilde{\sigma}_2 : \mathcal{M}^+(\theta_0, \theta_\infty) \rightarrow \mathcal{M}^+(\theta_0 + 1, \theta_\infty + 1)$. The formulas are given with respect to the coordinates $a = a_0, q$ of an open subset (namely $q \neq 0$ in the chart ST_2) of the first space and $\tilde{a} = \tilde{a}_0, \tilde{q}$ of

an open subset of the second space. The assumption that the operator $z\frac{d}{dz} + A(a, q, \theta_0, \theta_\infty, z)$, belonging to $\mathcal{M}^+(\theta_0, \theta_\infty)$, is equivalent, by a transformation of the type $U_{-2}z^{-2} + U_{-1}z^{-1} + U_0 + U_1z + U_2z^2$, to the operator $z\frac{d}{dz} + A(\tilde{a}, \tilde{q}, \theta_0 + 1, \theta_\infty + 1, z)$, belonging to $\mathcal{M}^+(\theta_0 + 1, \theta_\infty + 1)$, leads to the following formulas

$$\tilde{q} = \frac{-4q^2\theta_\infty + 4a^2 - 4q^3t - q^2t^2 - 4q^4 - 4q^2\theta_0 + \theta_0^2 - 4a\theta_0}{4q(qt - \theta_0 + 2a + 2q^2)}$$

$$\tilde{a} = \frac{long}{16q^2(qt - \theta_0 + 2a + 2q^2)^2}.$$

The substitution $a = q' + \frac{1}{2}$ in the first formula produces \tilde{q} in terms of q, q' and the parameters θ_0, θ_∞ , this is *the Bäcklund transformation in terms of solutions*. The second formula is obtained by substitution $\tilde{a} = \tilde{q}' + \frac{1}{2}$ and an expression for \tilde{q}' coming from the first formula and the equation for q'' .

The term $qt - \theta_0 + 2a + 2q^2$ in the denominator of the formulas indicates that the morphism $\tilde{\sigma}_3$ is in general a rational equivalence and is not defined on leaves of the foliation with $a = q' + \frac{1}{2}$ and $q' + q^2 + \frac{t}{2}q + \frac{-\theta_0+1}{2} = 0$. This occurs precisely when $\theta_0 = -\theta_\infty$ and the reducible locus of $\mathcal{M}^+(\theta_0, \theta_\infty)$ is not present in the corresponding $\mathcal{M}^+(1 + \theta_0, 1 + \theta_\infty)$ (compare Observations 2.3 and the Riccati equations for reducible families).

We note that the group $\langle \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3 \rangle$ contains the two shifts $\theta_0 \mapsto \theta_0 + 2$, $\theta_\infty \mapsto \theta_\infty$ and $\theta_0 \mapsto \theta_0$, $\theta_\infty \mapsto \theta_\infty + 2$. One observes that, in comparison with the book of Gromak et al. [Gr] and Okamoto's paper [O3], there is still a missing generator for the group of all Bäcklund transformations of PIV. This generator does not seem to come from a 'natural' automorphism of \mathbf{S}^+ (i.e., constructions of linear algebra for differential modules and operations with the differential field $\mathbb{C}(z)$). In the final section we will investigate another set of differential modules \mathbf{S} , inspired by Noumi's symmetric form of PIV ([No, No-Y]). As is shown by Noumi, this will easily produce all Bäcklund transformations and moreover all rational and Riccati solutions.

3 The Noumi–Yamada family

M. Noumi and Y. Yamada produced a 3×3 -Lax pair, arising from the Lax formalism of the modified KP hierarchy, for the symmetric form of PIV,

namely

$$z \frac{d}{dz} + \begin{pmatrix} \epsilon_1 & f_1 & 1 \\ z & \epsilon_2 & f_2 \\ f_0 z & z & \epsilon_3 \end{pmatrix}, \quad \frac{d}{dt} + \begin{pmatrix} -q_1 & 1 & 0 \\ 0 & -q_2 & 1 \\ z & 0 & -q_3 \end{pmatrix}, \text{ leading to equations}$$

$$\epsilon'_1 = \epsilon'_2 = \epsilon'_3 = 0; \quad f_1 - f_2 = -q_1 + q_3; \quad f_2 - f_0 = q_1 - q_2; \quad f_0 - f_1 = q_2 - q_3, \\ f'_0 = f_0(f_1 - f_2) + (1 - \epsilon_1 + \epsilon_3); \quad f'_1 = f_1(f_2 - f_0) + (\epsilon_1 - \epsilon_2); \quad f'_2 = f_2(f_0 - f_1) + (\epsilon_2 - \epsilon_3).$$

Since the local exponents ϵ_* at $z = 0$ are constants in an isomonodromic family, we can and will suppose $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$. Then we may and will also suppose that $q_1 + q_2 + q_3 = 0$. Further $t = f_0 + f_1 + f_2$ is assumed.

Then f_1 satisfies $y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 - 2ty^2 + (\frac{t^2}{2} + \theta_\infty)y - \frac{(\theta_0 - 1)^2}{2y}$ with $\theta_0 = 1 + \epsilon_1 - \epsilon_2$, $\theta_\infty = 1 + \epsilon_1 - \epsilon_3$. After rescaling $t \mapsto \frac{t}{\sqrt{2}}$, $y \mapsto -\sqrt{2}y$ one obtains 'our' equation $y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + ty^2 + (t^2 + 4\theta_\infty)\frac{y}{8} - \frac{(\theta_0 - 1)^2}{8y}$.

Using this symmetric form one finds the extended Weyl group of A_2 as group of Bäcklund transformations. For example $\pi : (f_0, f_1, f_2) \mapsto (f_1, f_2, f_0)$ translates into the 'missing' Bäcklund transformation of §2.3, namely

$$\theta_0 \mapsto -\frac{1}{2}\theta_0 + \frac{1}{2}\theta_\infty + 2, \quad \theta_\infty \mapsto -\frac{3}{2}\theta_0 - \frac{1}{2}\theta_\infty + 2.$$

This is the inspiration for the new class **S** of differential modules M over $\mathbb{C}(z)$, defined by: $\dim M = 3$; $\Lambda^3 M$ is trivial; the only singular points are $0, \infty$; 0 is regular singular and the Katz invariant of ∞ is $\frac{2}{3}$. After scaling the variable z the generalized eigenvalues at ∞ are:

$$q_0 = z^{2/3} + \frac{t}{3}z^{1/3}, \quad q_1 = \zeta^2 z^{2/3} + \zeta \frac{t}{3}z^{1/3}, \quad q_2 = \zeta z^{2/3} + \zeta^2 \frac{t}{3}z^{1/3} \text{ where } \zeta = e^{2\pi i/3}.$$

Invariant lattices at $z = \infty$. For $M \in \mathbf{S}$ the operator $D = \nabla_{z \frac{d}{dz}}$ has at $z = \infty$ has the form $z \frac{d}{dz} + \text{diag}(q_0, q_1, q_2)$ with respect to a basis e_0, e_1, e_2 . A lattice Λ at $z = \infty$ is called *invariant* if $z^{-1}D(\Lambda) \subset \Lambda$. We respect to the basis $h_0 = e_0 + e_1 + e_2$, $h_1 = z^{1/3}(e_0 + \zeta e_1 + \zeta^2 e_2)$, $h_2 = z^{-1/3}(e_0 + \zeta^2 e_1 + \zeta e_2)$,

D has the form $z \frac{d}{dz} + \begin{pmatrix} 0 & z & \frac{t}{3} \\ \frac{t}{3} & \frac{1}{3} & 1 \\ z & \frac{t}{3}z & -\frac{1}{3} \end{pmatrix}$. Thus $\Lambda_0 := \langle h_0, h_1, h_2 \rangle$ is an

invariant lattice. $\Lambda_1 := \langle h_0, z^{-1}h_1, h_2 \rangle$ is the only invariant lattice of codimension 1 in Λ_0 and the operator D has with respect to this basis the form

$$z \frac{d}{dz} + \begin{pmatrix} 0 & 1 & \frac{t}{3} \\ \frac{t}{3}z & -\frac{2}{3} & z \\ z & \frac{t}{3} & -\frac{1}{3} \end{pmatrix}. \text{ The invariant lattice } \Lambda_2 := \langle z^{-1}h_0, z^{-1}h_1, h_2 \rangle$$

has codimension 2 in Λ_0 . All invariant lattices are $\{z^n \Lambda_i \mid n \in \mathbb{Z}, i = 0, 1, 2\}$.

The Noumi–Yamada Lax pair has one additional feature, namely: there exists $U \in \mathrm{GL}(3, \mathbb{C}[[z]])$ with $U = 1 + U_1 z + U_2 z^2 + \dots$ such that

$$U^{-1} \left\{ z \frac{d}{dz} + \begin{pmatrix} \epsilon_1 & f_1 & 1 \\ z & \epsilon_2 & f_2 \\ f_0 z & z & \epsilon_3 \end{pmatrix} \right\} U = z \frac{d}{dz} + \begin{pmatrix} \epsilon_1 & * & * \\ 0 & \epsilon_2 & * \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \text{ with all } * \in \mathbb{C}.$$

Level structure for \mathbf{S} . This leads to a ‘level structure’ or ‘parabolic structure’ for the elements $M \in \mathbf{S}$ consisting of differential submodules $M_1 \subset M_2 \subset \mathbb{C}((z)) \otimes M$ of dimensions 1 and 2 over $\mathbb{C}((z))$. Let \mathbf{S}^+ denote the set of the differential modules in \mathbf{S} , provided with a level structure.

The moduli space \mathcal{R} for the analytic data.

The singular directions d for $q_k - q_\ell$ are computed as follows. $z \frac{d}{dz}(y) = (q_k - q_\ell)y$ has solution $\exp(\frac{3}{2}(\zeta^{2k} - \zeta^{2\ell})z^{2/3} + 3(\zeta^k - \zeta^\ell)z^{1/3})$. Write $z = e^{id}$ and $\zeta^{2k} - \zeta^{2\ell} = |\zeta^{2k} - \zeta^{2\ell}|e^{i\phi(k, \ell)}$. Then $|y(re^{id})|$ has maximal descent for $r \rightarrow \infty$ if and only if $\frac{2}{3}d + \phi(k, \ell) = \pi + \mathbb{Z}2\pi$ (or $d = \frac{3}{2}\pi - \frac{3}{2}\phi(k, \ell) + \mathbb{Z}3\pi$).

k	ℓ	ϕ	d
0	1	$\frac{1}{6}\pi$	$\frac{5}{4}\pi$
1	0	$\frac{7}{6}\pi$	$\frac{11}{4}\pi$
0	2	$\frac{11}{6}\pi$	$\frac{7}{4}\pi$
2	0	$\frac{5}{6}\pi$	$\frac{1}{4}\pi$
1	2	$\frac{9}{6}\pi$	$\frac{9}{4}\pi$
2	1	$\frac{3}{6}\pi$	$\frac{3}{4}\pi$

The analytic data consists of the formal monodromy and the six Stokes matrices at $z = \infty$. The product of the formal monodromy and the Stokes matrices for the singular directions $d \in [0, 2\pi)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is equal to the topological monodromy $M(x_*) := \begin{pmatrix} x_4 & 0 & x_1 x_4 + 1 \\ 1 & 0 & x_1 \\ x_3 & 1 & x_1 x_3 + x_2 \end{pmatrix}$. Its characteristic polynomial is $\lambda^3 - (x_2 + x_4 + x_1 x_3)\lambda^2 + (-x_1 - x_3 + x_2 x_4)\lambda - 1$.

The moduli space \mathcal{R} for the analytic data consists of the tuples $x_* = (x_1, \dots, x_4)$ since the other two Stokes matrices can be expressed in the Stokes matrices for the singular directions in $[0, 2\pi)$. Thus $\mathcal{R} \cong \mathbb{A}^4$. The elements of the parameter space \mathcal{P} are the sets of eigenvalues of the topological monodromy. Thus $\mathcal{P} = \{\lambda^3 - e_1\lambda^2 + e_2\lambda - 1 \mid e_1, e_2 \in \mathbb{C}\}$. Let $\mathcal{R}(P)$ be the fibre above $P \in \mathcal{P}$. If P has three distinct roots, then $\mathcal{R}(P)$ is a smooth surface. If P has roots a, a, a^{-2} with $a \neq a^{-2}$, then the point $x_* \in \mathcal{R}(P)$ with $x_* \neq (-a^{-1}, a, -a^{-1}, a)$ is regular and $M(x_*)$ has two Jordan blocks. The point $(-a^{-1}, a, -a^{-1}, a) \in \mathcal{R}(P)$ is singular and has type A_1 . Further $M(-a^{-1}, a, -a^{-1}, a)$ has three Jordan blocks. If P has roots a, a, a (and thus $a^3 = 1$), then the point $x_* \in \mathcal{R}(P)$ with $x_* \neq (-a^{-1}, a, -a^{-1}, a)$ is regular and $M(x_*)$ has one Jordan block. The point $x_* = (-a^{-1}, a, -a^{-1}, a)$ is singular and has type A_2 . The matrix $M(-a^{-1}, a, -a^{-1}, a)$ has two Jordan blocks.

Level structure for \mathcal{R} and \mathcal{P} . For an element of \mathcal{R} we introduce a ‘level structure’ which consists of subspaces $L_1 \subset L_2 \subset \mathbb{C}^3$ of dimensions 1 and 2 which are invariant under the topological monodromy (at $z = \infty$ or, equivalently, at $z = 0$). The corresponding space is denoted by \mathcal{R}^+ . The level structure for a $P \in \mathcal{P}$ consists of a tuple (μ_1, μ_2, μ_3) with $\mu_1\mu_2\mu_3 = 1$ and $P = \prod_{j=1}^3 (\lambda - \mu_j)$. The corresponding space is denoted by \mathcal{P}^+ . The morphism $par : \mathcal{R}^+ \rightarrow \mathcal{P}^+$ is defined by $((x_*), L_1, L_2) \mapsto (\mu_1, \mu_2, \mu_3)$, where μ_1 is the eigenvalue of $M(x_*)$ on L_1 and μ_2 is that of $M(x_*)$ on L_2/L_1 .

One observes that \mathcal{R}^+ is the closed subspace of $\mathbb{C}_{x_*}^4 \times \mathbb{P}_{y_*}^2 \times (\mathbb{P}^2)_{z_*}^* \times \mathcal{P}^+$, where $\mathcal{P}^+ = \{(\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3 \mid \mu_1\mu_2\mu_3 = 1\}$, given by the equations:

$$M(x_*)y = \mu_1 y, \quad y := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad zM(x_*) = \mu_3 z, \quad z := (z_1, z_2, z_3), \quad \sum y_j z_j = 0.$$

Indeed, $\mathbb{C}y$ and the kernel of $z \in (\mathbb{C}^3)^*$ are the $M(x_*)$ -invariant spaces $L_1 \subset L_2 \subset \mathbb{C}^3$. Further $par : \mathcal{R}^+ \rightarrow \mathcal{P}^+$ is the projection onto the last factor. The fibre $\mathcal{R}^+(\mu_*)$ of par above the point $(\mu_*) \in \mathcal{P}^+$ maps to the fibre $\mathcal{R}(P)$ of $\mathcal{R} \rightarrow \mathcal{P}$ above the point $P = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)$.

Proposition 3.1 *$res : \mathcal{R}^+(\mu_*) \rightarrow \mathcal{R}(P)$ is the minimal resolution of $\mathcal{R}(P)$.*

A straightforward computation proves this statement. In particular, the fibre of res above a non singular point is just one point since there is only one level structure possible. If two of the μ_* are equal, then the preimage under res of the singular point is a \mathbb{P}^1 , consisting of the lines $\mathbb{C}y$ in the two-dimensional

eigenspace for a and the kernel of z is this two-dimensional eigenspace. If the three μ_* are equal, then the preimage under res of the singular point is *a pair of intersecting projective lines*. In this case the Jordan form of $M(x_*)$ has two blocks, $\mathbb{C}y$ is a line in the two-dimensional eigenspace of \mathbb{C}^3 , $\mathbb{C}z$ is a line in the two-dimensional eigenspace of the dual $(\mathbb{C}^3)^*$ and $\sum y_j z_j = 0$.

Proposition 3.2 *The natural maps $\mathbf{S} \rightarrow \mathcal{R} \times T$ and $\mathbf{S}^+ \rightarrow \mathcal{R}^+ \times T$ (with $T = \mathbb{C}$) are bijections.*

Proof. In the first case one applies [vdP-Saito], Theorem 1.7. The second case follows from the observation that the level structure $M_1 \subset M_2 \subset \mathbb{C}((z)) \otimes M$ induces subspaces $L_1 \subset L_2 \subset \mathbb{C}^3$ of dimensions 1 and 2, invariant under the topological monodromy, and visa versa. \square

The Noumi–Yamada moduli space $\mathcal{N}^+(\epsilon_)$.*

ϵ_* denotes a triple $(\epsilon_1, \epsilon_2, \epsilon_3)$ with $\sum \epsilon_j = 0$. The set $\mathbf{S}^+(\epsilon_*)$ consists of the tuples $(M, M_1 \subset M_2) \in \mathbf{S}^+$ such that $M_1 = \mathbb{C}((z))b_1$ with $\delta_M(b_1) = \epsilon_1 b_1$ and $M_2/M_1 = \mathbb{C}((z))b_2$ with $\delta_M(b_2) = \epsilon_2 b_2$. Let \mathcal{V} denote the free bundle on \mathbb{P}^1 of rank 3.

The points of the moduli space $\mathcal{N}^+(\epsilon_*)$ are the isomorphy classes of connections $D := \nabla_{z \frac{d}{dz}} : \mathcal{V} \rightarrow \mathcal{O}([\infty]) \otimes \mathcal{V}$ with a level structure consisting of D -invariant submodules $V_1 \subset V_2 \subset \widehat{\mathcal{V}}_0$ of rank 1 and 2 such that $\widehat{\mathcal{V}}_0/V_j$, $j = 1, 2$ have no torsion and such that there is a tuple $(M, M_1 \subset M_2) \in \mathbf{S}^+(\epsilon_*)$ with M is the generic fibre of \mathcal{V} , $\mathbb{C}((z)) \otimes V_j = M_j$, $j = 1, 2$ and $\widehat{\mathcal{V}}_\infty$ is the lattice $\Lambda_0 \subset \mathbb{C}((z^{-1})) \otimes M$.

Proposition 3.3 *$\mathcal{N}^+(\epsilon_*)$ is the affine space \mathbb{A}^3 with coordinates f_0, f_1, f_2 ,*

$t = f_0 + f_1 + f_2$ and the connection is represented by $z \frac{d}{dz} + \begin{pmatrix} \epsilon_1 & f_1 & 1 \\ z & \epsilon_2 & f_2 \\ f_0 z & z & \epsilon_3 \end{pmatrix}$.

Proof. The level structure provides $H^0(\mathcal{V})$ with a basis e_1, e_2, e_3 such that $D = z \frac{d}{dz} + A_0 + A_1 z$ with traceless constant matrices A_0, A_1 and $A_0 = \begin{pmatrix} \epsilon_1 & * & * \\ 0 & \epsilon_2 & * \\ 0 & 0 & \epsilon_3 \end{pmatrix}$. This is unique up to the action of $B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$. The lattice condition at $z = \infty$ is equivalent to $U \{ z \frac{d}{dz} + A_0 + A_1 z \} U^{-1} =$

$$z \frac{d}{dz} + \begin{pmatrix} 0 & z & \frac{t}{3} \\ \frac{t}{3} & \frac{1}{3} & 1 \\ z & \frac{t}{3}z & -\frac{1}{3} \end{pmatrix} \text{ for some } U = U_0(1 + U_{-1}z^{-1} + \dots) \in \mathrm{GL}_3(\mathbb{C}[[z^{-1}]]).$$

This is equivalent to the equations $A_1 = U_0^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & \frac{t}{3} & 0 \end{pmatrix} U_0$ and $A_0 =$

$$U_0^{-1} \begin{pmatrix} 0 & 0 & \frac{t}{3} \\ \frac{t}{3} & \frac{1}{3} & 1 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} U_0 + [A_1, U_{-1}].$$
 A MAPLE computation produces matrices

$$U_0 \text{ and } U_{-1} \text{ such that } A_0 = \begin{pmatrix} \epsilon_1 & * & * \\ 0 & \epsilon_2 & * \\ 0 & 0 & \epsilon_3 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ * & 1 & 0 \end{pmatrix}.$$

Moreover U_0 is unique up to multiplication by a non zero constant and U_{-1} is unique up to adding a matrix V with $[A_1, V] = 0$. Thus we found a representation of the connection in the ‘Noumi–Yamada form’ and this form is unique with respect to the action of the Borel group B on \mathcal{V} . Therefore the Noumi–Yamada form represents the moduli space $\mathcal{N}^+(\epsilon_*)$. \square

The map $\mathcal{N}(\epsilon_*) \rightarrow \mathbf{S}^+(\epsilon_*)$ is injective and not bijective. This is due to the choice of a free vector bundle \mathcal{V} in the construction of $\mathcal{N}(\epsilon_*)$. The aim is to avoid this choice and to construct a smooth partial completion $\widehat{\mathcal{N}}(\epsilon_*)$ such that $\widehat{\mathcal{N}}(\epsilon_*) \rightarrow \mathbf{S}^+(\epsilon_*)$ is bijective. As in Section 2, this will imply that the extended Riemann–Hilbert map $\widehat{\mathcal{N}}(\epsilon_*) \rightarrow \mathcal{R}^+(\mu_*) \times T$ (with $\mu_j = e^{2\pi i \epsilon_j}$ for $j = 1, 2, 3$) is an analytic isomorphism. Moreover $\widehat{\mathcal{N}}(\epsilon_*)$ is the Okamoto–Painlevé space and $\mathcal{R}^+(\mu_*)$ is the space of the initial conditions.

Construction of $\widehat{\mathcal{N}}(\epsilon_)$.*

The points of $\widehat{\mathcal{N}}(\epsilon_*)$ are (the isomorphism classes of) the tuples $(\mathcal{V}, D, L_1, L_2)$ with $D = \nabla_{z \frac{d}{dz}} : \mathcal{V} \rightarrow O([\infty]) \otimes \mathcal{V}$ is a connection on a vector bundle \mathcal{V} of rank 3. *We require the following:* The connection $\widehat{\mathcal{V}}_\infty$ is isomorphic to Λ_1 . In other terms $\widehat{\mathcal{V}}_\infty$ has a basis over $\mathbb{C}[[z^{-1}]]$ for which D has the form

$$z \frac{d}{dz} + \begin{pmatrix} 0 & 1 & \frac{t}{3} \\ \frac{t}{3}z & -\frac{2}{3} & z \\ z & \frac{t}{3} & -\frac{1}{3} \end{pmatrix}.$$
 Further $L_1 = \mathbb{C}[[z]]Y$ is a subconnection of $\widehat{\mathcal{V}}_0$

such that $\widehat{\mathcal{V}}_0/L_1$ has no torsion and $DY = \epsilon_1 Y$. Further $L_2 = \mathbb{C}[[z]]Z$ is a subconnection of $\widehat{\mathcal{V}}_0^*$, the dual of $\widehat{\mathcal{V}}_0$, such that $\widehat{\mathcal{V}}_0^*/L_2$ has no torsion and $DZ = \epsilon_3 Z$. Moreover $\Lambda^3(\widehat{\mathcal{V}}_0)$ is trivial and $L_2(L_1) = 0$.

The map $F : \widehat{\mathcal{N}}(\epsilon_*) \rightarrow \mathbf{S}^+(\epsilon_*)$, sends $(\mathcal{V}, D, L_1, L_2)$ to its generic fibre M together with the level structure on $\mathbb{C}((z)) \otimes M$ obtained from L_1, L_2 . Conversely, for a given element $(M, M_1 \subset M_2) \in \mathbf{S}^+(\epsilon_*)$ one defines the connection (\mathcal{V}, D) with generic fibre M , by prescribing $\widehat{\mathcal{V}}_\infty \cong \Lambda_1$. The additional data L_1, L_2 imply that $\widehat{\mathcal{V}}_0$ is represented by $z \frac{d}{dz} + \begin{pmatrix} \epsilon_1 & * & * \\ 0 & \epsilon_2 & * \\ 0 & 0 & \epsilon_3 \end{pmatrix}$ with all $*$ in $\mathbb{C}[[z]]$. This implies that F is bijective in the following cases:

- μ_1, μ_2, μ_3 are distinct;
- $\mu_1 = \mu_2 \neq \mu_3$ and $\epsilon_2 - \epsilon_1 \geq 0$;
- $\mu_1 = \mu_3 \neq \mu_2$ and $\epsilon_3 - \epsilon_1 \geq 0$;
- $\mu_1 \neq \mu_2 = \mu_3$ and $\epsilon_3 - \epsilon_2 \geq 0$;
- $\mu_1 = \mu_2 = \mu_3$ and $\epsilon_2 - \epsilon_1, \epsilon_3 - \epsilon_2 \geq 0$.

In the sequel we will only consider these cases.

In order to give $\widehat{\mathcal{N}}(\epsilon_*)$ the structure of an algebraic variety we observe that \mathcal{V} has degree -1 and type $O \oplus O \oplus O(-1)$ since (\mathcal{V}, D) is irreducible. We identify \mathcal{V} with $Oe_1 \oplus Oe_2 \oplus O(-[\infty])e_3$. The matrix of D with respect to the basis e_1, e_2, e_3 has trace zero and is denoted by $\begin{pmatrix} a_0 + a_1 z & a_2 + a_3 z & a_4 + a_5 z + a_6 z^2 \\ a_7 + a_8 z & a_9 + a_{10} z & a_{11} + a_{12} z + a_{13} z^2 \\ a_{14} & a_{15} & a_{16} + a_{17} z \end{pmatrix}$. The generator $Y = \sum_{n \geq 0} Y_n z^n$ of L_1 with $Y_n = Y_n(1)e_1 + Y_n(2)e_2 + Y_n(3)e_3$ is unique up to multiplication by a constant. The generator $Z = \sum_{n \geq 0} Z_n z^n$ of L_2 with $Z_n = Z_n(1)e_1^* + Z_n(2)e_2^* + Z_n(3)e_3^*$ is unique up to multiplication by a constant. The $Y_*(*)$, $Z_*(*)$ are regarded as homogeneous coordinates.

The space \mathcal{A} is defined by the indeterminates $a_*, Y_*(*), Z_*(*)$ and the relations induced by the above requirements. We note that for given ϵ_* , such that the above restrictions are satisfied, the $Y_n(*), Z_n(*)$ are for $n \geq 1$ eliminated by the relations. Thus \mathcal{A} is an algebraic variety.

The group G of the automorphisms of \mathcal{V} act upon \mathcal{A} . By construction, the set theoretic quotient $\mathcal{A}(\mathbb{C})/G$ coincides with $\mathbf{S}^+(\epsilon_*)$. Thus the analytic map $R : \mathcal{A} \rightarrow \mathcal{R}^+(\mu_*) \times T$, where $\mu_j = e^{2\pi i \epsilon_j}$ for $j = 1, 2, 3$, is surjective and $R(\xi_1) = R(\xi_2)$ if and only if there is a $g \in G$ with $g\xi_1 = \xi_2$.

A long MAPLE session verifies: \mathcal{A} has a smooth geometric quotient by G . This quotient is by definition $\widehat{\mathcal{N}}(\epsilon_*)$ and the extended Riemann–Hilbert map $\widehat{\mathcal{N}}(\epsilon_*) \rightarrow \mathcal{R}^+(\mu_*) \times T$ is an analytic isomorphism.

References

- [Gr] V.I. Gromak ; I. Laine; S. Shimomura, *Painlevé differential equations in the complex plane*. de Gruyter Studies in Mathematics, 28. Walter de Gruyter & Co., Berlin, 2002.
- [In] M. Inaba, *Moduli of parabolic connections on a curve and Riemann-Hilbert correspondence*, Preprint, 2006, [arXiv:math/0602004](#).
- [IIS1] M. Inaba, K. Iwasaki, and M.-H. Saito, *Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI. I*, Publ. Res. Inst. Math. Sci. **42** (2006), no. 4, 987–1089.
- [IIS2] M. Inaba, K. Iwasaki, and M.-H. Saito, *Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI. II*, Moduli spaces and arithmetic geometry (Tokyo), Adv. Stud. Pure Math., vol. 45, Math. Soc. Japan, Tokyo, 2006, pp. 387–432.
- [IISA] M. Inaba, K. Iwasaki and M.-H. Saito, *Dynamics of the sixth Painlevé Equations*, Théories Asymptotiques et Équations de Painlevé, Angers, Juin, 2004, "Séminaires et Congrès" of the Societe Mathematique de France (SMF)14, 2006, 103–167.
- [JMU] M. Jimbo, T. Miwa and K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ -function*, Physica D, **2**, (1981), 306–352.
- [JM] M. Jimbo and T. Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II.*, Physica D, **2**, (1981), 407–448.
- [No] M. Noumi, *Painlevé equations through symmetry*. Translated from the 2000 Japanese original by the author. Translations of Mathematical Monographs, 223. American Mathematical Society, Providence, RI, 2004
- [No-Y] M. Noumi and Y. Yamada, *Symmetries in the fourth Painlevé equation and Okamoto polynomials*. Nagoya Math. J. 153 (1999), 5386.
- [OO] Y. Ohyama, S. Okumura, *A coalescent diagram of the Painlevé equations from the viewpoint of isomonodromic deformations*. J. Phys. A 39 (2006), no. 39, 12129–12151.
- [O1] K. Okamoto, *Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé, Espaces des conditions initiales*, Japan. J. Math. **5**, (1979), 1–79.

- [O2] K. Okamoto, *Isomonodromic deformation and Painlevé equations and the Garnier system*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **33**, (1986), 575–618.
- [O3] K. Okamoto, *Studies on the Painlevé Equations III Second and Fourth Painlevé Equations, P_{II} and P_{IV}* , Math. Ann. 275, 221–255, 1986
- [O4] K. Okamoto, *The Hamiltonians associated to the Painlevé equations. The Painlevé property*, 735–787, CRM Ser. Math. Phys., Springer, New York, 1999.
- [vdP1] M. van der Put, *Families of linear differential equations and the Painlevé equations*, SMF, Séminaires & Congrès **27**. 2012, p.203–220.
- [vdP2] M. van der Put, *Families of linear differential equations related to the second Painlevé equation* in: Algebraic Methods in Dynamical Systems, Banach Center Publications, Volume 94, 2011, p. 247–262.
- [vdP-Sa] M. van der Put and M-H. Saito, *Moduli spaces for linear differential equations and the Painlevé equations.*, Ann. Inst. Fourier **59** (2009), no. 7, 2611–2667
- [vdP-Si] M. van der Put, M.F. Singer, *Galois Theory of Linear Differential Equations*, Grundlehren der mathematischen Wissenschaften, Volume 328, Springer Verlag 2003.
- [STT] M.-H. Saito, T. Takebe, H. Terajima, *Deformation of Okamoto-Painlevé pairs and Painlevé equations*, J. Algebraic Geom. **11** (2002), no. 2, 311–362.
- [S-Ta] Saito, M-H. and Takebe, T., *Classification of Okamoto-Painlevé pairs*, Kobe J. Math. 19 (2002), no. 1-2, 21–50.
- [SU] M.-H. Saito, H. Umemura, *Painlevé equations and deformations of rational surfaces with rational double points*. Physics and combinatorics 1999 (Nagoya), 320–365, World Sci. Publishing, River Edge, NJ, 2001.
- [STe] M.-H. Saito, H. Terajima, *Nodal curves and Riccati solutions of Painlevé equations*, J. Math. Kyoto Univ. **44**, (2004), no. 3, 529–568.
- [Sakai] H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. **220** (2001), 165–229.
- [T] H. Terajima, *Families of Okamoto-Painlevé pairs and Painlevé equations*, Ann. Mat. Pura Appl. (4), **186** (2007) no. 1, 99–146.